SINGULARIZING CARDINALS

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Abstract. We survey methods and applications of singularizing cardinals. The main ways of singularizing a cardinal is by Priky forcing, Namba forcing, Woodin tower. We survey what properties of these forcings, especially Prikry type posets, can be abstractly formalized without making specific assumptions on the forcing. Then we describe some applications, focusing on square properties. These show some limitations on ways to get failure of square properties together with violating SCH. We also discuss what happens to cofinalities when collapsing cardinals.

1. Introduction

Singularizing cardinals has a key role in infinitary combinatorics. For example, obtaining models of the failure of the singular cardinal hypothesis (SCH) usually is done by adding a lot of Cohen subsets to a large cardinal, and then singularizing it. On the other hand, it is hard to combine failure of SCH with compactness properties at the first successor of a singular. Compactness is the phenomenon that if a given property holds at every substructure of smaller cardinality of some object, then if holds for the whole object. This motivates the following question.

Question 1. *Suppose that $V \subset W$ are transitive models of set theory, κ is regular in V and a singular cardinal in W.*

1. *How close is W to a Prikry type forcing extension?*
2. *What can we say about compactness properties at $(\kappa^+)^W$? In which cases do we have square sequences at κ in W?*

Prikry forcing is one of the standard ways to singularize cardinals: given a measurable cardinal κ and a normal measure U on it, Prikry forcing with respect to U adds a generic cofinal sequence $\langle \alpha_n \mid n < \omega \rangle$ through κ, with the following characterization of genericity:

$$\forall A \in U, \text{ for all large } n, \alpha_n \in A.$$

Similar characterization of genericity holds in the uncountable cofinality case, when adding a Magidor sequence.

In the next section we will survey some results which approximate this phenomenon, but without making any assumptions about forcing extensions. It turns out that, under some reasonable assumptions, in the situation of

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the above question, there are ω-sequences through κ in W, that can meet many club subsets on a tail end. Here meeting clubs takes the place of meeting measure one sets, since we do not want to make any assumptions about W being a forcing extension. We use the term "pseudo Prikry" to describe these sequences because their properties approximate the genericity of a Prikry sequence, without referring to any measure.

Another way of singularizing cardinals is by a stationary Woodin tower forcing. A third way to singularize cardinals is by Namba forcing. In the basic case, forcing with Namba over a ground model V makes $(\aleph_2)^V$ have cofinality ω, while preserving $(\aleph_1)^V$. In the generic extension both $(\aleph_2)^V$ and $(\aleph_3)^V$ are collapsed.

A typical example of compactness is failure of square. The square principle was isolated by Jensen in his fine structure analysis of L. Square principles hold in models that sufficiently resemble L. On the other hand they are at odds with reflection properties and fail above large cardinals. There is tension between failure of SCH and failure of the weaker square properties. The reason is that it is difficult to avoid weaker square principles at singular cardinals. We will discuss applications of singularizing to square in section 3.

We will use the following notation: $\mathrm{cof}(\tau)$ and $\mathrm{cof}(<\tau)$ denote points of cofinality τ and less than τ, respectively. Similarly, we write $\mathrm{cof}^W(\tau)$ to denote the points of cofinality τ in W. For a set C, $\lim(C)$ is the set of all limit points of C and $\mathrm{nacc}(C)$ is the non-limit points of C. Also, we say that cardinal is collapsed to mean that it is no longer a cardinal in the outer model. Otherwise, it is preserved.

2. Pseudo Prikry sequences and more

By arguments in Gitik [5] and independently in Džamonja-Shelah [4], if $V \subset W$, and κ is an inaccessible cardinal in V and a singular cardinal in W and $(\kappa^+)^V$ is preserved, then for every family $\langle C_\alpha \mid \alpha < \kappa^+\rangle \in V$ of clubs in κ^+, there is a sequence $\langle \delta_n \mid n < \omega\rangle$ cofinal in κ, such that for each α, for all large n, $\delta_n \in C_\alpha$. Moreover, for every $\lambda < \kappa$, it can be arranged that the cofinality of the δ_n's is above λ.

In [9] this theorem was generalized:

Theorem 2.1. *Suppose $V \subset W$ and $\kappa < \nu$ are cardinals in V, such that, ν is regular in V,*

- $(\omega_1)^W < \kappa$, $\mathrm{cf}^W(\kappa) = \omega$,
- *for all V-regular cardinals, τ, $\kappa \leq \tau \leq \nu$, $\mathrm{cf}^W \tau = \omega$, and*
- $\mu := (\nu^+)^V$ *reamins a cardinals in W.*

Suppose that $\langle D_\alpha \mid \alpha < \mu\rangle \in V$ is a sequence of club subsets of ν. Let $\lambda < \kappa$. Then in W there is a sequence $\langle \delta_n \mid n < \omega\rangle$ cofinal in ν, with each $\mathrm{cf}^W(\delta_n) \geq \lambda$, such that for every $\alpha < \mu$, for all large n, $\delta_n \in D_\alpha$.

The proof relies on the following fact due to Shelah, whose proof can be found in [10], Theorem 2.14.

Fact 2.2. Suppose that $\nu > \kappa > \aleph_1$ are cardinals, where ν is regular. Then there is a sequence $\langle S_\delta \mid \delta < \nu\rangle$ of stationary subsets of ν^+, such that $\bigcup_{\delta<\nu} S_\delta = \nu^+ \cap \operatorname{cof}(<\kappa)$ and each S_δ carries a partial square sequence $\langle C^\delta_\alpha \mid \alpha \in S_\delta \cap \mathrm{Lim}\rangle$. More precisely:

- each C^δ_α is a club subset of α with $o.t.(C^\delta_\alpha) < \kappa$, and
- if $\beta \in \lim(C^\delta_\alpha)$, then $\beta \in S_\delta$ and $C^\delta_\alpha \cap \beta = C^\delta_\beta$.

Recall the characterization of genericity for Prikry forcing with respect to a normal measure U: a sequence $\langle \alpha_n \mid n < \omega\rangle$ is generic if and only if for all measure one sets $A \in U$, for all large n, $\alpha_n \in U$. The above theorem provides an abstraction of that notion. Since we do not want to assume anything about a forcing or measures, we are looking at sequences meeting club subsets instead on a tail end. We call such sequences *pseudo Prikry*. Below we give a similar abstraction of super compact Prikry. Super compact Prikry with respect to a normal measure U on $\mathcal{P}_\kappa(\kappa^{+m})$ adds a generic ω-sequence through $\mathcal{P}_\kappa(\kappa^{+m})$. And actually, $\langle x_n \mid n < \omega\rangle$ is a geniric sequence if and only if for every $A \in U$, for all large n, $x_n \in A$. The next theorem abstracts this notion, by considering club subsets of $\mathcal{P}_\kappa(\kappa^{+m})$:

Below $V \subseteq W$ are two transitive models of set theory. When we use some set theoretic terminology or notation like "a regular cardinal", δ^+ etc. we shall mean it in the sense of V, unless otherwise stated.

Theorem 2.3. *([9]) Suppose that κ is a regular cardinal in V, and $m < \omega$ is such that in W $\kappa, \kappa^+, \dots \kappa^{+m}$ all have cofinality ω, $(\omega_1)^W < \kappa$, and $(\kappa^+)^W = \kappa^{+m+1}$. In V let $\langle D_\alpha \mid \alpha < \kappa^{+m+1}\rangle$ be a sequence of clubs of $P_\kappa(\kappa^{+m})$. Then in W there is a sequence $\langle x_n \mid n < \omega\rangle$ of elements of $P_\kappa(\kappa^{+m})$, such that for every $\alpha < \kappa^{+m+1}$ for large enough $n < \omega$, $x_n \in D_\alpha$.*

Moreover, if $\lambda < \kappa$ is an uncountable regular cardinal in W, we can assume that for $n < \omega$ and $k \leq m$, $\mathrm{cf}^W(\chi_{x_n}(\kappa^{+k})) \geq \lambda$.

In [9] it was asked whether the above can be generalized to infinite gaps. It turns out that the answer is yes. Moti Gitik in [6] showed the existence of a pseudo Prikry sequence for meeting clubs in $\mathcal{P}_\kappa(\tau)$, in the case of $V \subset W$, $V \models \kappa < \tau$ are both regular, and in W, $\tau = \bigcup_n z_n$, where each $z_n \in (\mathcal{P}_\kappa(\tau))^V$ (and some mild additional cardinal arithmetic assumptions). This generalizes Theorem 2.3. Below is a summary of his results.

Theorem 2.4. *(Gitik [6]) Suppose that $V \subset W$ are transitive models of set theory, and κ is regular uncountable in V, such that $\mu = (\tau^+)^V = (\kappa^+)^W$ and (†)$W \models \tau = \bigcup_n x_n$, each $x_n \in \mathcal{P}_\kappa(\tau)$. Then for every family $\langle C_\alpha \mid \alpha < \mu\rangle \in V$ of clubs subsets of $\mathcal{P}_\kappa(\tau)$, there is a pseudo Prikry sequence.*

Remark 1. (1) The assumption that τ is regular in V is essential. If τ is singular of cofinality less than κ, Gitik showed that this always will fail, just like it does in the model in [7].

(2) Note that the assumption above (†) implies that all V-regular cardinals in the interval $[\kappa, \tau]$ are singularized to have countable cofinality in W. This for example, will not be the case in a Namba type forcing extension.

(3) If $\tau < \kappa^{+\omega_1}$, then the converse also holds, as shown below.

Lemma 2.5. *(Gitik [6]) Suppose that $V \subset W$ are transitive models of set theory. Suppose that κ is a regular cardinal in V, and for some countable δ, $\mu := (\kappa^+)^W < (\kappa^{+\omega_1})^V$, and also that every V-regular τ in the interval $[\kappa, \mu)$ is singularized to have cofinality ω in W. Then for every $\delta < \omega_1$, such that $\kappa^{+\delta} < \mu$, there is $\langle x_n \mid n < \omega \rangle \in W$, such that every $x_n \in (\mathcal{P}_\kappa(\kappa^{+\delta}))^V$, such that $(\kappa^{+\delta})^V = \bigcup_n x_n$.*

Proof. By induction on δ. If $\delta = 0$, then it is clear. Suppose that $\delta = \eta + 1$ and the conclusion hold for η. I.e. $(\kappa^{+\eta})^V = \bigcup_n x_n$ for some $x_n \in (\mathcal{P}_\kappa(\kappa^{+\eta}))^V$. In W, let $(\kappa^{+\delta})^V = \sup_n \alpha_n$, with each $|\alpha_n|^V = \kappa^{+\eta}$. Suppose that this is witnessed by bijections $g_n : \kappa^{+\eta} \to \alpha_n$. Let $y_{n,m} = g_n{}'' x_m$. Then each $y_{n,m} \in (\mathcal{P}_\kappa(\kappa^{+\delta}))^V$, and $(\kappa^{+\delta})^V = \bigcup_{n,m} y_{n,m}$.

Next, suppose that δ is limit. I.e. $\delta = \sup_n \delta_n$ and by induction, for each n there is a sequence $\langle x^n_k \mid k < \omega \rangle$ of elements in $(\mathcal{P}_\kappa(\kappa^{+\delta_n}))^V$, such that $(\kappa^{+\delta_n})^V = \bigcup_k x^n_k$. Let $y_m = \bigcup_{n,k<m} x^n_k$, which is in $(\mathcal{P}_\kappa(\kappa^{+\delta}))^V$. Then $\bigcup_m y_m = \bigcup_{n,k<\omega} x^n_k = (\kappa^{+\delta})^V$. □

In particular if every V-regular cardinal between $[\kappa, \kappa^{+n}]$ is singularized to countable cofinality, then $(\kappa^{+n})^V$ can be written as the union of ω many elements in $(\mathcal{P}_\kappa(\kappa^{+n}))^V$. Same if every V-regular cardinal between $[\kappa, \kappa^{+\omega}]$ is singularized to countable cofinality.

The existence of pseudo Prikry sequences was generalized to the case for uncountable cofinalities by Lambie-Hanson [8]. In that paper he also generalized some of the above theorems to the case when the new successor of κ was a successor of a singular in V, and to the case of existence of *diagonal* pseudo-Prikry sequences.

In the above results, the recurring assumption is that regular cardinals in some interval $[\kappa, \nu]$, ν regular, are singularized to the same cofinality. By a theorem of Shelah, the last such cardinal (i.e. ν) will always have the same cofinality as its cardinality:

Theorem 2.6. *(Shelah) Suppose that $V \subset W$ are transitive models of set theory, ν is a regular cardinal in V, which is collapsed in W, so that $(\nu^+)^V$ is preserved. Then $W \models \mathrm{cf}(\nu) = \mathrm{cf}(|\nu|)$.*

Proof. Let $\mu := (\nu^+)^V$, and let $\langle A_\alpha \mid \alpha < \mu \rangle \in V$ be a family of almost disjoint unbounded subsets of ν. I.e. for every $\beta < \mu$, there are $\delta_\alpha < \kappa$, for all $\alpha < \beta$, such that $\langle A_\alpha \setminus \delta_\alpha \mid \alpha < \beta \rangle$ are pairwise disjoint. Such a sequence is called *an almost disjoint sequence* and it always exists for a regular cardinal ν.

Let $\kappa := |\nu|^W$ and let $g : \kappa \to \nu$ be a bijection. Suppose for contradiction that in W, $\mathrm{cf}(\nu) \neq \mathrm{cf}(\kappa)$. Then for all $\alpha < \mu$, there is $\tau_\alpha < \kappa$, such that $g"\tau_\alpha$ is unbounded in A_α. Since μ remains a regular cardinal in W, the map $\alpha \mapsto \tau_\alpha$ is constant on an unbounded set. Let $I \subset \mu$ be unbounded, and $\tau < \kappa$ be such that for all $\alpha \in I$, $\tau = \tau_\alpha$. Let $\beta < \mu$, be such that $W \models |I \cap \beta| > \tau$. We can choose such a point since in W, $|I| = \kappa^+$. Let $\langle \delta_\alpha \mid \alpha < \beta \rangle$ be such that setting $B_\alpha := A_\alpha \setminus \delta_\alpha$, we have that $\langle B_\alpha \mid \alpha < \beta \rangle$ are pairwise disjoint.

But then since $g"\tau$ is unbounded in each B_α for $\alpha \in I$, we have that $\langle g"\tau \cap B_\alpha \mid \alpha \in I \cap \beta \rangle$ are more than $|\tau|$ many pairwise disjoint non empty subsets of $g"\tau$. Contradiction.

□

It is an old open problem whether this is true when ν is singular in V. In that case there is not always an almost disjoint sequence of unbounded subsets of ν. We will discuss this more in the last section. It is also open whether cardinals in a finite interval $[\kappa, \kappa^{+n}]$ can be singularized to nonuniform cofinalities.

3. Consequences on square properties

In this section we survey some application of the above theorems on singularizing cardinals and square properties. This is especially useful if we want to understand how much compactness can hold in the presence of failure the singular cardinal hypothesis (SCH). Recall that SCH holds at a singular strong limit cardinal κ if $2^\kappa = \kappa^+$. More generally, for a singular κ, SCH at κ is the statement that if $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. The standard way of violating SCH involves singularizing a cardinal. So, we are especially interested in what square sequences are added then.

Cummings and Schimmerling showed in [3] that after Prikry forcing at κ, $\square_{\kappa,\omega}$ holds in the generic extension.

Let us recall some definitions. Square at κ, $\square_\kappa$, states that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that each C_α is a club subset of α, $o.t.(C_\alpha) \leq \kappa$, and if $\delta \in \lim C_\alpha$, then $C_\alpha \cap \delta = C_\delta$. There are various weakenings allowing multiple guesses for the clubs at each point. More precisely, the principle $\square_{\kappa,\lambda}$ states that there is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$, such that:

(1) $1 \leq |\mathcal{C}_\alpha| \leq \lambda$,
(2) every $C \in \mathcal{C}_\alpha$ is a club in α, $o.t.(C) \leq \kappa$,
(3) if $C \in \mathcal{C}_\alpha$ and $\delta \in \lim(C)$, then $C \cap \delta \in \mathcal{C}_\delta$.

Note that $\square_{\kappa,1}$ is just the usual square at κ, and $\square_{\kappa,\kappa}$ is weak square at κ, $\square^*_\kappa$. By the results in Gitik [5] and independently in Džamonja-Shelah [4], it was implicit that if $V \subset W$ are transitive class models of ZFC such that κ is an inaccessible cardinal in V, singular of countable cofinality in W, and $(\kappa^+)^V = (\kappa^+)^W$, then $W \models \square_{\kappa,\omega}$. The conclusion still holds even if we collapse some cardinals above κ:

Theorem 3.1. *([9]) Suppose $V \subset W$ and $\kappa < \nu$ are such that, ν is a regular cardinal in V, κ is a singular cardinal in W of countable cofinality, for all V-regular cardinals τ, with $\kappa \leq \tau \leq \nu$, $W \models \mathrm{cf}(\tau) = \omega$, and $(\nu^+)^V = (\kappa^+)^W$. Then in W, $\square_{\kappa,\omega}$ holds.*

Proof. Work in V. Denote $\mu := \nu^+$. Let χ be some big enough cardinal, and $<_\chi$ be a well-ordering of H_χ. For $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$, let $\langle M^\alpha_\delta \mid \delta < \nu\rangle$ be a continuous $\subset$-increasing sequence of elementary submodels of $\langle H_\chi, \in, <_\chi ...\rangle$, such that:

(1) $\alpha, \kappa \in M^\alpha_0$,
(2) for each $\delta < \nu$, $|M^\alpha_\delta| < \nu$,
(3) for each $\delta < \nu$, $M^\alpha_\delta \cap \nu \in \nu$.

Claim 3.2. *For all $\alpha < \mu$, $\delta < \nu$ with $\mathrm{cf}(\delta) > \omega$, if $M^\alpha_\delta \cap \alpha$ is cofinal in α, then $M^\alpha_\delta \cap \alpha$ is ω-closed.*

Proof. Otherwise, let $\langle \beta_i \mid i < \omega\rangle$ be an increasing sequence of points in $M^\alpha_\delta \cap \alpha$, such that $\beta := \sup_i \beta_i \notin M^\alpha_\delta \cap \alpha$. Let β^* be the least ordinal in $M^\alpha_\delta \cap \alpha$ above β. If $\lambda := \mathrm{cf}(\beta^*) < \nu$, then $\lambda \in M^\alpha_\delta \cap \nu \in \nu$. Then $\lambda + 1 \subset M^\alpha_\delta$, and so M^α_δ is cofinal in β^*, which is a contradiction.

It follows that the cofinality of β^* must be ν. Say $M^\alpha_\delta \models h : \nu \to \beta^*$ is cofinal. For every i, let $\beta^*_i \in \mathrm{ran}(h) \setminus \beta_i$. The order type of $\mathrm{ran}(h)$ is $\nu \cap M^\alpha_\delta$. And $\mathrm{cf}(\nu \cap M^\alpha_\delta) = \mathrm{cf}(\delta) > \omega$. So there is something in the range of h above β. Contradiction with the choice of β^*.

$\square$

For $\alpha < \mu$ define $D_\alpha := \{\sup(M^\alpha_\delta \cap \nu) \mid \delta < \nu\}$. We claim that this is club in ν. D_α is closed by construction since the sequence of models is continuous. It is unbounded, since $\nu \subset \bigcup_\delta M^\alpha_\delta$. Note that this also implies that $\alpha \subset \bigcup_\delta M^\alpha_\delta$, since the $<_\chi$-least function in H_χ from ν onto α is in each M^α_δ.

By Theorem 2.1 there is a sequence $\langle \lambda_i \mid i < \omega\rangle \in W$, such that for all large i, $\lambda_i \in D_\alpha$ and for all i, $\mathrm{cf}^W(\lambda_i) > \omega$. For $\alpha < \mu$ with $\mathrm{cf}^V(\alpha) < \kappa$, define $\mathcal{C}_\alpha := \{\overline{M^\beta_\delta \cap \alpha} \mid \alpha \leq \beta < \mu, \delta < \nu, M^\beta_\delta$ is cofinal in β and $\alpha, (\exists i) M^\beta_\delta \cap (\nu)^V = \lambda_i\}$. Here $\overline{M^\beta_\delta \cap \alpha}$ denotes the closure of $M^\beta_\delta \cap \alpha$. Otherwise, if $\mathrm{cf}^V(\alpha) = \kappa$, and so $\mathrm{cf}^W(\alpha) = \omega$, we let $\mathcal{C}_\alpha$ be a singleton of some ω-sequence cofinal in α.

Claim 3.3. $1 \leq |\mathcal{C}_\alpha| \leq \omega$.

Proof. Each $\mathcal{C}_\alpha$ is nonempty because for all large i, $\lambda_i \in D_\alpha$. Now suppose $\beta, \beta', \delta, \delta'$ are such that:

- $M^\beta_\delta \cap \nu = M^{\beta'}_{\delta'} \cap \nu = \lambda_i$,
- $M^\beta_\delta, M^{\beta'}_{\delta'}$ are both cofinal in α.

We claim that $M^\beta_\delta \cap \alpha = M^{\beta'}_{\delta'} \cap \alpha$. If $\mathrm{cf}(\alpha) = \omega$, then $\alpha \in M^\beta_\delta, M^{\beta'}_{\delta'}$. By elementarity, the $\prec_\chi$-least function from ν onto α is in $M^\beta_\delta \cap \nu = M^{\beta'}_{\delta'} \cap \nu$, and the result follows. Now suppose that $\mathrm{cf}(\alpha) > \omega$. Since $\mathrm{cf}(\delta) = \mathrm{cf}(\delta') = \mathrm{cf}(\lambda_i) > \omega$, then $M^\beta_\delta \cap \alpha, M^{\beta'}_{\delta'} \cap \alpha$ are both ω-clubs in α, and so $M^\beta_\delta \cap M^{\beta'}_{\delta'} \cap \alpha$ is an ω-club in α. By the above for any $\eta \in M^\beta_\delta \cap M^{\beta'}_{\delta'} \cap \alpha$, $M^\beta_\delta \cap \eta = M^{\beta'}_{\delta'} \cap \eta$, and so $M^\beta_\delta \cap \alpha = M^{\beta'}_{\delta'} \cap \alpha$.

□

Next we use $\langle \mathcal{C}_\alpha \mid \alpha < \mu \rangle$ to obtain a $\square_{\kappa,\omega}$ sequence. In W fix is a sequence $\langle F_\beta \mid \beta < \nu \rangle$, such that each F_β is a club in β of order type at most κ, and for $\delta \in \lim(F_\beta)$, $F_\beta \cap \delta = F_\delta$. Enumerate $\mathcal{C}_\alpha := \{C^\alpha_n \mid n < \omega\}$ and $C^\alpha_n = \{\gamma^{\alpha,n}_\xi \mid \xi < \nu^\alpha_n\}$, where $\nu^\alpha_n = o.t.(C^\alpha_n) < \nu$. Define $E^\alpha_n := \{\gamma^{\alpha,n}_\xi \mid \xi \in F_{\nu^\alpha_n}\}$ and let $\mathcal{E}_\alpha = \{E^\alpha_n \mid n < \omega\}$. Then $\langle \mathcal{E}_\alpha \mid \alpha < \mu \rangle$ is a $\square_{\kappa,\omega}$ sequence. □

On the other hand, Gitik and Sharon [7] showed that failure of weak square at a singular is consistent with the failure of SCH. They constructed a forcing where κ is singularized, $(\kappa^{+\omega+1})^V$ becomes the successor of κ, and weak square of κ fails in the generic extension. So the assumption that $\mu := (\kappa^+)^W$ was a successor of a V-regular cardinal is necessary. What about when μ is inaccessible in V? In that case, by standard arguments one can usually define a weak square sequence.

Proposition 3.4. *Suppose $V \subset W$ and $\kappa < \mu$ are cardinals in both V, W, such that, $V \models \mu$ is inaccessible, κ is regular; and $W \models \mathrm{cf}(\kappa) = \omega, \kappa^+ = \mu$, and for all V-regular cardinals τ, with $\kappa \leq \tau < \mu$, $W \models \mathrm{cf}(\tau) = \omega$. Then in W, $\square^*_\kappa$ holds.*

Proof. For $\alpha < \mu$ with $\mathrm{cf}^V(\alpha) < \kappa$, let $\mathcal{C}_\alpha$ be the set of all clubs $C \subset \alpha$, such that $C \in V$, $o.t.(C) < \kappa$. Since μ is inaccessible, $|\mathcal{C}_\alpha| < \mu$. And for $\alpha < \mu$ with $\mathrm{cf}^V(\alpha) = \kappa$, and so $\mathrm{cf}^W(\alpha) = \omega$, let $\mathcal{C}_\alpha$ be a singleton containing some ω-cofinal sequence through α (in W). Then $\langle \mathcal{C}_\alpha \mid \alpha < \mu \rangle$ is a weak square sequence at κ. □

Woodin asked whether one can violate SCH at $\aleph_\omega$ and at the same time have failure of weak square at $\aleph_\omega$. As pointed out above, Gitik and Sharon showed that this is possible for some singular cardinal. Then they bought down their result to $\aleph_{\omega^2}$:

Theorem 3.5. *(Gitik-Sharon [7]) Suppose that in V, κ is supercompact, $\mu := \kappa^{+\omega+1}$. Then there is a forcing extension W, where $\kappa = \aleph_{\omega^2}$, $\mu = \aleph_{\omega^2+1}$, SCH fails at κ and weak square at κ also fails.*

It is still open whether the above can be achieved for $\aleph_\omega$, and answer Woodin's original question. By the results discussed above, if one starts with a regular (and possibly large) cardinal κ in the ground model, and singularizes it, then the new successor of κ should have been a successor of a singular in the ground model. More precisely:

Corollary 3.6. *Suppose $V \subset W$, κ is a regular cardinal in V, a singular cardinal in W, $\mu = (\kappa^+)^W$, all V-regular cardinals in the interval $[\kappa, \mu)$ are have cofinality ω in W, and $\neg\square^*_\kappa$ in W. Then μ must be the successor of a singular cardinal in V.*

4. Open problems

Many of the theorems on pseudo Prikry sequences do generalize for uncountable cofinalities. But it remains open whether applications on square properties as in Theorem 3.1 hold for uncountable cofinalities. For example, in the simplest case, one can ask:

Question 2. *Suppose that $V \subset W$, κ is inaccessible in V and singular in W with $\mathrm{cf}^W(\kappa) = \tau > \omega$, $(\kappa^+)^V = (\kappa^+)^W$. Does $\square_{\kappa,\tau}$ hold in W?*

The main obstacle here would be to deal with the points whose V-cofinality is κ when trying to build a square sequence. In the countable cofinality case, one can take an ω sequence, and since there are no limit points, that suffices. We note that in the uncountable case, using the methods of Theorem 3.1, one can build a partial $\square_{\kappa,\mathrm{cf}(\kappa)}$ sequence in the outer model, over the set of points of V-cofinality less than κ.

Question 3. *Suppose $V \subset W$ are models of set theory. If κ is regular in V, and in W, κ is a singular cardinal of cofinality ω, $(\kappa^{+n+2})^V = (\kappa^+)^W$, then what is the cofinality of $(\kappa^+)^V, ..., (\kappa^{+n})^V$ in W?*

By Shelah, we know the cofinality of the last cardinal collapsed, $(\kappa^{n+1})^V$, will be ω in W. And with supercompact Prikry forcing we can arrange $V \subset W$, where all of the cardinals $(\kappa^+)^V, ..., (\kappa^{+n})^V$ will have countable cofinality in W. So the question is, is this always the case? By lemma 2.5, this is equivalent to asking whether κ^{+n+1} is the union of a sequence $\langle x_n \mid n < \omega\rangle$, where each $x_n \in (\mathcal{P}_\kappa(\kappa^{+n+1}))^V$.

Next, we turn to the problem mentioned at the end of Section 2.

Question 4. *Can we have models $V \subset W$, such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$?*

Let us first note that if such a pair $V \subset W$ exist, CH must fail in W, since setting $\nu := (\aleph_\omega)^V$, we have $(\aleph_2)^W = \mu \leq \nu^\omega = \aleph_1^\omega = 2^\omega$, as computed in W. Moreover, if in V there is an almost disjoint sequence of unbounded subsets of $\aleph_\omega$, then the same proof at in Shelah's theorem will work to give a negative answer. Note that the existence of such a sequence follows from square. So a positive answer would require large cardinals. By Cummings [1] a positive answer would imply that there is *a bad scale* in V at $\aleph_\omega$ and that W cannot be a $(\aleph_{\omega+1})^V$-c.c. extension of V. More generally, he showed that if $V \subset W$ are such that $\aleph^V_{\omega+1} = \aleph^W_{n+1}$ for some $n > 0$, then in V there is a stationary $S \subset \aleph_{\omega+1}$ of bad points of cofinality ω_n. By pcf facts almost every point in $\aleph_{\omega+1}$ of cofinality $\geq \omega_4$ is good ([11]). It follows that $\aleph_{\omega+1}$ cannot be turned into $\aleph_5$ or more. The following remain open:

Question 5. (1) *Can we have a stationary set of bad points of cofinality ω_2, or of cofinality ω_3?*
(2) *Is it true that almost all points cofinality $\geq \omega_4$ are approachable (which is a strengthening of good points)?*

We finish with the following conjecture by Moti Gitik [6]:

Conjecture: Suppose that $V \subset W$ have the same cardinals on a final segment, and κ is a regular cardinal in V, singular with cofinality ω in W, and $(\aleph_1)^V = (\aleph_1)^W$. Then for some intermediate model $V \subset V' \subset W$ with $V' \models \mathrm{cf}(\kappa) = \omega$, V' is a forcing extension of V by either Prikry, Namba, or Woodin tower forcing.

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